Word of Mass: The Relationship between Mass Media and Word-of-Mouth

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Preliminary version

March 15, 2017

Abstract

This paper studies the optimal pricing and advertising strategies of a firm in the presence of word-of-mouth communication. In the model, a monopolist produces new product and chooses the price and amount of advertising. Consumers can learn about the product directly from advertising or from their neighbors, who have acquired it. We show that the optimal advertising level is a non-monotone function in the network connectivity. An increase in connectivity facilitates diffusion and consequently raises the payoff to advertising. However, when connectivity becomes sufficiently high, its further increase leads to a congestion effect and the optimal advertising level falls. We also show that an increase in the advertising cost may actually lead to a higher consumer surplus.

JEL Classification numbers: D42, D83, D85
Keywords: word-of-mouth, advertising, diffusion, networks, pricing

1 Introduction

This paper studies the relationship between advertising and word-of-mouth communication. Both advertising and word-of-mouth increase consumers' awareness of the product, but act quite differently. Although, an advertising is costly activity, a producer can freely choose its amount. In contrast, word-of-mouth depends on the consumers behavior and network structure and can be affected only in the indirect way. In the paper we show that

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depending on the parameters of the model these two phenomena may compliment each other or serve as substitutes.

In the model a firm creates new product and sells it to a continuum of consumers. We assume that the product quality is revealed after the development process and producer treats it as given exogenously. Consumers are embedded into a social network, which is represented by a generalized random graph[1]. Each consumer has an outside option distributed according to a uniform distribution and buys the product if the utility of purchase is higher than the outside option.

Initially, consumers are not aware of the product and to induce sales the innovator advertises the product directly to some proportion of consumers. Advertising is costly and has diminishing returns to scale. The rest of the population can learn about the product and its quality from their neighbors. We assume that consumers tell their friends about the product only if they buy it and thus find information worth to spread. In the model there is no asymmetry of information and everyone who becomes aware of the product immediately knows its quality. One of the explanations may be that nowadays consumers find all relevant information in the internet. The producer knows statistical properties of consumer network and chooses the amount of advertising and pricing strategy to maximize profits.

We show that for moderate levels of connectivity, product sales are non-monotone function in the cost of advertising. More precisely, as cost increases sales of the product first decrease, but after some threshold level sales increase. An increase in the cost of advertising lowers the optimal advertising level. The producer partially offsets this by reducing the price that facilitates word-of-mouth communication. These two effects work in the opposite directions.

In the beginning when cost of advertising is sufficiently low, the majority of consumers become aware of the product through mass media. The diffusion in this case looks like a great number of small interconnected islands emerged around consumers who got an advertisement. The amoun of word-of-mouth is limited and the first effect dominates. However, when advertising cost becomes sufficiently the producer partially offsets the drop in advertising by lowering the price. The product diffusion looks like few big islands and a small decrease in the price generates a sizeable cascade of sales. This happens since the price is already low and majority of consumer neighbors are unaware of the product. We show that in this case the indirect effect dominates and sales increase.

Considering welfare implications, we show that both consumer surplus and social welfare first decrease in advertising cost, but then increase. As we know, for a sufficiently low advertising cost its increase lowers sales. At the same time the producer lowers the price to stimulate word-of-mouth. However, the price drop is not enough to offset decrease in sales, since for a sufficiently low cost word-of-mouth plays minor role in information

[1]Generalized random graph is a graph selected with a uniform probability from a set of graphs that obey given statistical properties. In our case all graphs in the set have some specified degree distribution.

diffusion. As cost of advertising grows further at some point sales start to increase. In this case both effects work in the same direction, and more consumers buy the product at a lower price. Thus for a sufficiently high advertising cost consumer surplus increases. Moreover, for a sufficiently large cost, an increase in the consumer surplus becomes large enough to compensate the decrease in the producer surplus and total welfare increases.

We also show that the advertising level is a non-monotone function in the network connectivity. More precisely, first the amount of advertising increases in the connectivity, but after a threshold value decreases. When the connectivity is sufficiently low, each advertisement generates small cascade of sales as there are few channels for the information to spread on. As a result producer chooses low advertising level. A growth in the connectivity increases efficiency of advertising and the optimal amount of advertising increases. As the connectivity grows further, both the amount of advertising and the average size of sales cascade generated by an advertisement increase. In this case a further increase in the connectivity leads to a congestion effect, when some part of recommendations are made to consumers who already are aware of the product. The diffusion slows down and the payoff on the advertising decreases. The producer lowers advertising and substitutes it with a higher word-of-mouth by lowering the price.

To our best knowledge this is the first paper that studies the interaction of word-of-mouth and advertising in the explicit form, assuming non-trivial network structure. The previous papers that study diffusion of word-of-mouth such as Lopez-Pintado (2008), Chuhay (2013) assume that only an infinitesimal part of population receives advertising. The rest of population can find out about the product only by means of word-of-mouth. In contrast, in our paper the advertising level is firm's choice variable, which is affected by the network structure.

The most related paper to our work is Campbell (2012), which considers the optimal pricing strategy of a monopolist in the presence of word-of-mouth communication. The main result of this paper regarding advertising is that the price elasticity of demand decreases in the advertising level. In our paper, we make the advertising level endogenous variable. We show that word-of-mouth and advertising behave as substitutes if we consider a change in the advertising cost, but may show complementarity when we vary network connectivity. Using numerical analysis Campbell (2012) shows that consumer surplus may increase in the advertising cost. In our paper, we identify conditions under which consumer surplus is increasing in advertising cost. Moreover, we show that an increase in the advertising cost may be beneficial for the society as whole.

Galeotti and Goyal (2009) studies the model of strategic diffusion of information, where authors allow for network externalities in adoption decision. In the paper the authors limit diffusion process only to immediate neighbors of a consumer who receives information. This assumption may play a crucial role, especially in the case when individual's adoption decision depends on the adoption ratio[2]. In contrast, we model diffusion process in the

[2]As authors note, an increase in the number of contacts negatively affects the probability of product

explicit way, which allows us to study the effects of average connectivity on the optimal pricing and advertising strategies.

The rest of the paper is organized as follows. Section 2 describes the model and demand function is derived. Section 3 presents the main results regarding the impact of advertising cost and network connectivity on the optimal price, sales, advertising level and social welfare. Section 4 concludes.

2 Model

There is a continuum of consumers that are embedded into a social network represented by a classical random graph with a given connectivity. A firm creates a new product, for which there are no close substitutes and acts as a monopolist on the market. We assume that product quality $v \in [0,1]$ is realized after a production process took place and the firm treats it as given exogenously. Observing quality of the product and knowing network connectivity, the firm chooses price P and the amount of advertising s to maximize profits.

Initially, consumers are not aware of the product. To start sales the company advertises the product to the population. With probability s each consumer receives an advertising. With complementary probability $1-s$ a consumer does not get the advertising and may receive information about the product only from her neighbors, who already have acquired the product. The advertising is costly and producer pays $\frac{c}{1-s}$ for advertising the product to proportion s of consumers. The cost function is convex in s, which represents the idea that it is impossible to control who gets an advertisement. Thus to reach an increasing part of consumers the amount of advertising should grow exponentially.

All consumers have outside option γ_i, which is distributed according to uniform distribution $U[0,1]$. A consumer i buys the product if the valuation of the product purchase $v-P$ is higher than her outside option γ_i. Thus a randomly selected consumer buys the product with probability $q = v - P$. Once a consumer buys the product, all her neighbors become aware of it and may buy it too.

In the model we consider an equilibrium state where diffusion already has taken place and the demand is given by the number of purchases that consumers made. The diffusion stops when all consumers who learn about the product do not buy it or do not have neighbors.

2.1 Demand function

With probability s a randomly chosen consumer gets an advertisement directly from the producer and buys the product with probability $v-P$. With probability $1-s$ the consumer does not get advertising and the only way for her to find out about the product is to hear

adoption and impede diffusion. However, there is also additional effect. Once a product is adopted by a consumer more neighbors become aware of it.

about it from neighbors. Let's assume that a randomly selected neighbor of a consumer buys the product with probability $\hat{w}$. Thus a consumer with k links does not hear about the product if no one of her neighbors buys it, which happens with probability $(1-\hat{w})^k$. With complementary probability $1-(1-\hat{w})^k$ at least one of consumer's neighbors buys the product and consumer learns about the product. Since, a randomly selected consumer has k links with probability $p(k)$, in expected terms she hears about the product and buy it with probability $(v-P)\sum_{k=0}^{\infty} p(k)(1-(1-\hat{w})^k)$. Thus the demand function is the following expression:

$$D(s,v,P) = s(v-P) + (1-s)(v-P)\sum_{k=0}^{\infty} p(k)(1-(1-\hat{w})^k)$$
$$= (v-P)\left(1-(1-s)\sum_{k=0}^{\infty} p(k)(1-\hat{w})^k\right)$$

$$D_A(s,v,P) = s(w-P)+(1-s)(w-P)\sum_{k=0}^{\infty} p(k)\sum_{j=0}^{k}\rho^j(1-\rho)^{k-j}\left(1-(1-\hat{w}_A)^j-(1-\hat{w}_B)^{k-j}\right)$$
$$= (w-P)\left(1-(1-s)\sum_{k=0}^{\infty} p(k)\sum_{j=0}^{k}\rho^j(1-\rho)^{k-j}\left((1-\hat{w}_A)^j+(1-\hat{w}_B)^{k-j}\right)\right)$$
$$= (w-P)\left(1-(1-s)\sum_{k=0}^{\infty} p(k)\sum_{j=0}^{k}\rho^j(1-\rho)^{k-j}\left((1-\hat{w}_A)^j+(1-\hat{w}_B)^{k-j}\right)\right)$$

To close the model we should formulate a self-consistency condition for $\hat{w}$. In general, degree distribution of neighbor is different from the one of a randomly selected consumer. The more links a consumer has the greater is the probability that she is someone's neighbor. Thus a consumer with k links has k-times higher probability to be a neighbor of randomly selected consumer than a consumer with just one link. Therefore, the probability to have a neighbor with k links is proportional to $kp(k)$. After normalization we obtain a degree distribution of neighboring consumer $\xi(k)$, which is the following:

$$\xi(k) = \frac{kp(k)}{\sum_{j=1}^{\infty} jp(j)} = \frac{kp(k)}{z_1}.$$

A neighboring consumer can be reached through one of her links. Thus the probability that a neighbor hears about the product from someone else and buys it is given by $(v-P)\sum_{k=1}^{\infty}\xi(k)(1-(1-\hat{w})^{k-1})$. Thus the probability that a neighbor buys the product is the following:

$$\hat{w} = s(v-P) + (1-s)(v-P)\sum_{k=1}^{\infty}\xi(k)(1-(1-\hat{w})^{k-1})$$

$$= (v - P)\left(1 - (1-s)\sum_{k=1}^{\infty}\xi(k)(1-\hat{w})^{k-1}\right)$$

The producer maximizes profits choosing amount of advertisement s and price P by solving the following problem:

$$\max_{s,P}\ P(v-P)\left(1-(1-s)\sum_{k=0}^{\infty}p(k)(1-\hat{w}(s,v,P))^{k}\right) - \frac{c}{1-s}$$
$$s.t.\ \hat{w} = (v-P)\left(1-(1-s)\sum_{k=1}^{\infty}\xi(k)(1-\hat{w})^{k-1}\right)$$

In the following analysis we assume that the network is represented by a classical random graph and thus the degree distribution is Poisson $p(k) = \frac{\lambda^k e^{-\lambda}}{k!}$. In the case of Poisson degree distribution the distribution of links of neighboring node $\xi(k)$ equals to $p(k-1)$. In particular, it implies that probabilities to buy the product for a randomly chosen consumer w and a randomly selected neighbor $\hat{w}$ are the same. The substitution of Poisson degree distribution reduces the problem to the following form:

$$\max_{s,P}\ P(v-P)\left(1-(1-s)e^{-\lambda w}\right) - \frac{c}{1-s}$$
$$s.t.\ w = (v-P)\left(1-(1-s)e^{-\lambda w}\right) \tag{1}$$

The following Lemma presents the solution to the maximization problem in the implicit form.

Lemma 1 *Given that* $\max\{1, \lambda\sqrt{c}\} < e^{\frac{\lambda w^*}{2}} < \min\left\{\frac{1}{2}\left(\frac{v}{\sqrt{c}} + \lambda\sqrt{c}\right), e^{\frac{\lambda}{2}}\right\}$ *there is a unique interior solution to the maximization problem, which is given by the following equations:*

$$s^* = 1 - \frac{2\sqrt{c}e^{\frac{\lambda w^*}{2}} - \lambda c}{v};\ P^* = \frac{v}{2}\left(1 - \frac{\lambda\sqrt{c}}{2e^{\frac{\lambda w^*}{2}} - \lambda\sqrt{c}}\right),$$

where w^* *is the solution to* $w^* = \frac{v}{2} - \frac{\sqrt{c}}{2}\left(\frac{2}{e^{\frac{\lambda w^*}{2}}} - \frac{\lambda v}{2e^{\frac{\lambda w^*}{2}} - \lambda\sqrt{c}}\right)$.

Proof See Appendix □

Lemma 1 gives a solution for the optimal price and advertising as a function of w^*. Note that w^* is the solution to the transcendental equation, which does not have closed form solution. The immediate corollary from the previous lemma is a condition on the advertising cost, such that there is non-zero advertising in the equilibrium.

Corollary 1 *If advertising cost* c *is higher than* v^2 *then there is no advertising in the equilibrium.*

Corollary 1 states that there is upper limit for the advertising cost, such that for c higher than $\bar{c}$ there is no interior equilibrium for any parameters of the model.

3 Main results

In this section we consider the effect of advertising cost and network connectivity on the sales, optimal price and awareness of the product. In the case of full information all consumers are aware of the product and profit function has the following form $P(v-P)$. The optimal price in this case is $P^*_{FI} = \frac{v}{2}$ and in equilibrium sales are $\frac{v}{2}$. Despite the fact that consumers are not aware of the product sales in the case of limited awareness of consumers may be higher than in the case of full information.

Proposition 1 *For sufficiently high connectivity λ sales in the case of incomplete information are higher than in the case of complete information. The statement is also true for sufficiently high cost c and $\lambda v > 2$. Moreover, the optimal price is always lower than P^*_{FI}, the price in the case of full information.*

Proof See Appendix □

When consumers are not aware of the product the firm uses costly advertising and then relies on further word-of-mouth diffusion to inform consumers about the product. Perhaps surprisingly, Proposition 1 implies that when consumers are not aware of the product sales may be higher than in the case of full information. This is the case when cost of advertising or network connectivity are sufficiently high.

In the first case when advertising is quite costly, the firm relies mostly on the word-of-mouth diffusion, which crucially depends on the product price. Recall, that consumers spread information about the product further only if they buy it. Thus to offset low level of advertising the firm lowers the price. When the connectivity is sufficiently high, a price reduction has substantial impact on word-of-mouth diffusion. Thus in the case of high advertising cost the producer sets low price and actual sales become higher than in the case of full information. The same logic applies in the case of sufficiently high network connectivity. To use high spreading efficiency of the network the monopolist sets low price.

3.1 The impact of the advertising cost

One of the important characteristics of the diffusion process is a share of consumers who are aware of the product. Each consumer that becomes aware of the product buys it with probability $v - P^*$. Since the share of consumers that buy the product is given by w^* the proportion of consumers who know about the product is simply $\frac{\hat{w}^*}{v-P^*}$. The following proposition formulates the results regarding the optimal price, the amount of advertising, awareness of the product and diffusion perimeter.

Proposition 2 *The optimal price P^* and amount of advertising s^* decrease in the cost of advertising c. The same is true about awareness of the product and diffusion perimeter.*

Proof See Appendix □

The first result is quite straightforward. An increase in the advertising cost leads to a lower level of advertising. The second part of the result states that the optimal price falls in the advertising cost. We already know that the advertising level decreases in the cost. As advertising becomes costlier the producer substitutes it with word-of-mouth communication by lowering the price. Indeed, a price decrease makes the product attractive to a longer chains of buying consumers. An increase in word-of-mouth communication offsets the decrease in the advertising level. However, as the result implies word-of-mouth substitutes advertising only partially and overall awareness of the product falls.

Proposition 3 *In general, sales of the product is non-monotone function in advertising cost c. More precisely, if $1 < \lambda v < 4$ then sales of the product first decrease, but after some level increase in c. If $\lambda v < 1$ sales are decreasing in c on the whole range, while if $\lambda v > 4$ sales always increase in c.*

Proof See Appendix □

According to Proposition 2 an increase in the advertising cost lowers both advertising level and the price. If we consider sales these two effects work in the opposite directions. Proposition 3 states that for the intermediate values of λv sales are non-monotone in the cost. In the beginning when the cost is sufficiently small most consumers become aware of the product through the mass media. In this case the diffusion looks like a great number of interconnected small islands emerged around consumers who got the advertising.

When the cost of advertising is sufficiently high, the advertising level and total awareness of the product are quite low. According to Proposition 2 the price is low too. The diffusion now looks like few big islands. A purchase of the product by a consumer on the perimeter generates a sizeable cascade of sales. This happens since the price is quite low and majority of consumer's neighbors are not aware of the product. In this case, the indirect price effect dominates the direct effect of advertising costs and sales increase. The increasing part appears only when λv is higher than 1. If the opposite is true, whatever small is the price, the diffusion is limited and sales always decrease in the advertising cost.

An interesting question is how the advertising cost affects consumers' welfare. If consumer i buys the product instead of the outside option she gains $v - P^* - \gamma_i$. We know that a consumer buys the product only if γ_i is lower than $v - P^*$. Thus the change in the consumer surplus can be represented as the following:

$$w^* \int_0^{v-P^*} (v - P^* - \gamma) \frac{1}{v - P^*} d\gamma = w^* \frac{v - P^*}{2}$$

The following proposition relates consumer surplus to the advertising cost.

Proposition 4 *If $\lambda v > 1$ the consumer surplus is non-monotone functions in advertising cost c. More precisely, first consumer surplus falls, but after some level consumers become* *better-off as the cost increases. When $\lambda v < 1$ consumer surplus decreases in the cost on the whole range.*

Proof See Appendix □

We have seen that when advertising cost is sufficiently low, an increase in the cost lowers sales. At the same time the producer lowers the price to stimulate word-of-mouth communication. Initially, when advertising cost is sufficiently low, the price drop is not enough to offset decrease in sales, since word-of-mouth communication plays minor role in the information diffusion. Thus consumer surplus decreases in the advertising cost. However, for sufficiently high levels of advertising cost its further growth increases sales. In this case both effects work in the same direction as more consumers buy the product at a cheaper price. Thus when the advertising cost is sufficiently high consumer surplus increases in it.

The important point here is that consumer surplus increases only if $\lambda v > 1$ and thus even infinitesimal advertisement may lead to a purchase of the product by some non-zero share of the population. When $\lambda v < 1$ the decrease in the price has limited impact of sales and that is why sales and consumer surplus do not increase.

We have seen that consumers may benefit from an increase in the advertising cost. This happens because the producer by lowering the price substitutes word-of-mouth for advertising. The important question is whether taxation of advertising can be beneficial for the society as a whole. Total welfare consists of three parts. Producer surplus $P^* w^* - \frac{c+t}{1-s^*}$, consumer surplus $w^* \frac{v-P^*}{2}$ and gains from taxation $\frac{t}{1-s^*}$. Summing up we obtain:

$$SW = w^*(c+t)\frac{v + P^*(c+t)}{2} - \frac{c}{1-s^*}$$

Proposition 5 *If $2 < \lambda v < 6$ then social welfare first decreases in c up to the point where $c = \frac{1-\lambda v+\sqrt{1+4\lambda v}}{\lambda^2}$ and then increases. If $\lambda v < 2$ then social welfare always decreases while for $\lambda v > 6$ social welfare always increases in c.*

Proof See Appendix □

Thus if λv is sufficiently high an increase in the consumer surplus due to a lower price compensates fall in the profits of producer and total welfare increases. The result crucially relies on the sufficiently high connectivity of the consumer network which facilitates word-of-mouth spreading.

3.2 The impact of connectivity

In the previous section we have seen substitution effect between advertising and word-of-mouth. More precisely, when advertising cost increases the producer turns to a cheaper word-of-mouth communication by decreasing the price. In this section we study the effect of network connectivity on the optimal amount of advertising and price. In contrast, to previous result we show that advertising and word-of-mouth may compliment each other. At the beginning, when connectivity grows, word-of-mouth compliments advertising and both increase. However, as connectivity grows further, the relationship is reversed and word-of-mouth becomes a substitute for the advertising. The following propositions formalize the result.

Proposition 6 *When advertising cost c is sufficiently close to the upper limit, the amount of advertising is a non-monotone function in connectivity λ. More precisely, for sufficiently small λ the amount of advertising increases in λ, while for sufficiently high λ decreases. If the advertising cost is sufficiently low then the advertising level always decreases in λ.*

When connectivity is sufficiently low, each advertising generates small cascade of sales as there are few channels for the information to spread on. A growth in the connectivity increases spreading efficiency of the network and hence increases the payoff on advertising. Thus for a sufficiently low levels of the connectivity, word-of-mouth serves as a complement to the advertising and both move in the same direction with the connectivity.

However, at some point a further increase in the connectivity leads to a congestion effect, when some advertisements are received by consumers who already know about the product from their neighbors. The payoff on the advertising decreases and the producer switches from advertising to relatively more efficient word-of-mouth. One of the immediate corollaries is that the advertising level reaches its maximum for intermediate values of connectivity λ, when advertising cost c is sufficiently close to the upper limit.

Proposition 7 *The optimal price is a non-monotone function in connectivity λ. More precisely, first the price decreases in the connectivity, but after some point increases.*

The intuition behind the result of Proposition 7 is the same as in the case of relationship between advertising and λ. In both cases, an increase in the connectivity makes word-of-mouth and advertising more efficient, which leads to a decrease in the optimal price. However, at some point the congestion effect comes into play and crowds out both advertising and word-of-mouth, which leads to an increase in the optimal price.

Proposition 8 *If network connectivity λ is sufficiently small then both consumer surplus and sales increase in λ. When network connectivity is sufficiently high and advertising cost is sufficiently low both consumer surplus and sales decrease in λ.*

Probably surprisingly Proposition 8 states that a higher connectivity and thus higher spreading efficiency of the network is not always better for consumers. When the connectivity is sufficiently low information about the product is scarce. In this case an increase in the connectivity leads to a lower price and higher advertising level, which increases consumer surplus. However, when the connectivity is sufficiently high and advertising cost

c is sufficiently low, a further increase in the connectivity leads to a higher price. This happens since a consumer in one way or another becomes aware of the product that is why it does not pain to cut some channels by increasing the price.

4 Conclusion

This paper studies the relationship between advertising and word-of-mouth communication. Both advertising and word-of-mouth increase consumers' awareness of the product, but act differently. In the paper we show that depending on the parameters of the model these two phenomena may compliment each other or act as substitutes. In particular, we show that for a sufficiently low connectivity levels both word-of-mouth and advertising are complements and grow with the connectivity. However, when connectivity becomes sufficiently high the producer substitutes advertising with relatively more efficient word-of-mouth.

We show that an increase in the advertising cost may actually increase consumer surplus. When advertising cost and network connectivity are sufficiently high, word-of-mouth is relatively more efficient than advertising and producer reduces the price to increase word-of-mouth communication. This price drop turns out to be sufficient to offset lower advertising and as a result lower product awareness.

We find that the optimal advertising level is a non-monotone function in the network connectivity. An increase in the connectivity facilitates word-of-mouth communication and consequently raises the payoff on advertising. However, for a sufficiently high connectivity levels its further increase leads to a congestion effect, when some consumers receive multiple recommendations and the optimal advertising level falls.

In the further research we plan to study the effect of the product quality on the optimal advertising and pricing strategies. This will allow us to differentiate the optimal marketing strategies for goods of high and low quality and to confront our findings with the data.

References

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5 APPENDIX

Proof of Lemma 1

Taking the derivative with respect to s and P and expressing the derivative of w from the constraint we get:

$$\frac{P(v-P)}{e^{\lambda w}-\lambda(1-s)(v-P)}-\frac{c}{(1-s)^2}=0 \tag{2}$$

$$-\frac{e^{-\lambda w}\left(e^{\lambda w}-(1-s)\right)\left(-(v-2P)e^{\lambda w}+\lambda(1-s)(v-P)^2\right)}{e^{\lambda w}-\lambda(1-s)(v-P)}=0 \tag{3}$$

Equations (2), (3) and constraint (1) give us solution to the maximization problem. Solving (3) for s we get two solutions. The first solution $s=1-e^{\lambda w}$ implies zero diffusion. The second is the following:

$$s=1-\frac{(v-2P)e^{\lambda w}}{\lambda(v-P)^2} \tag{4}$$

Substituting (4) to (2) we get:

$$\frac{(v-P)^2e^{-2\lambda w}\left((v-2P)^2e^{\lambda w}-c\lambda^2(v-P)^2\right)}{(v-2P)^2}=0$$

Solving the last equation for P, we get two possible candidates for the solution: $P_1=\frac{v}{2}\left(1+\frac{1}{1+\frac{2e^{\frac{\lambda w}{2}}}{\sqrt{c}\lambda}}\right)$ and $P_2=\frac{v}{2}\left(1+\frac{1}{1-\frac{2e^{\frac{\lambda w}{2}}}{\sqrt{c}\lambda}}\right)$. The first one implies advertisement over 100%, the second gives us interior solution. Substituting P_2 into equation for s and then substituting both to the constraint we get:

$$s^*=1-\frac{2\sqrt{c}e^{\frac{\lambda w^*}{2}}-c\lambda}{v} \tag{5}$$

$$P^*=\frac{v}{2}\left(1-\frac{1}{\frac{2e^{\frac{\lambda w^*}{2}}}{\sqrt{c}\lambda}-1}\right) \tag{6}$$

$$w^*=\frac{v}{2}-\frac{\sqrt{c}}{2}\left(\frac{2}{e^{\frac{\lambda w^*}{2}}}-\frac{v}{\frac{2e^{\frac{\lambda w^*}{2}}}{\lambda}-\sqrt{c}}\right) \tag{7}$$

Note that by the definition $0<s^*<1$, $0<P^*<v$ and $0<w^*<1$. The first condition gives us $0<2e^{\frac{\lambda w^*}{2}}-\lambda\sqrt{c}<\frac{v}{\sqrt{c}}$. The second condition implies $\lambda\sqrt{c}<2e^{\frac{\lambda w^*}{2}}-\lambda\sqrt{c}$. Finally, the third condition implies $1 < e^{\frac{\lambda w^*}{2}} < e^{\frac{\lambda}{2}}$. Combining all three conditions and rearranging we get:

$$\max\{\lambda\sqrt{c}, 2-\lambda\sqrt{c}\} < 2e^{\frac{\lambda w^*}{2}} - \lambda\sqrt{c} < \min\left\{\frac{v}{\sqrt{c}}, 2e^{\frac{\lambda}{2}} - \lambda\sqrt{c}\right\} \tag{8}$$

Or we can get:

$$\max\{1, \lambda\sqrt{c}\} < e^{\frac{\lambda w^*}{2}} < \min\left\{\frac{1}{2}\left(\frac{v}{\sqrt{c}} + \lambda\sqrt{c}\right), e^{\frac{\lambda}{2}}\right\} \tag{9}$$

Or we can get:

$$\max\{\frac{1}{\sqrt{c}}, \lambda\} < \frac{e^{\frac{\lambda w^*}{2}}}{\sqrt{c}} < \min\left\{\frac{1}{2}\left(\frac{v}{c} + \lambda\right), \frac{e^{\frac{\lambda}{2}}}{\sqrt{c}}\right\} \tag{10}$$

Uniqueness of the solution for w

Lets denote by $f(w)$ the right hand side of (7). We know that $f(0) = \frac{v-\sqrt{c}(2-\lambda\sqrt{c})}{2-\lambda\sqrt{c}} = \frac{v}{2-\lambda\sqrt{c}} - \sqrt{c}$, which is positive whenever $0 < s^*(0)$ and $P^*(0) < v$. Taking the derivative of $f(w)$ we get:

$$\frac{1}{2}\sqrt{c}\lambda e^{-\frac{\lambda w}{2}}\left(1 - \frac{\lambda v}{\left(2-\sqrt{c}\lambda e^{-\frac{\lambda w}{2}}\right)^2}\right)$$

Note that by (4) for s to be lower than 1, price P should be lower than $\frac{v}{2}$. Taking into account (10) we can conclude that $\sqrt{c}\lambda e^{-\frac{\lambda w}{2}} < 2$ for any w. Thus the last expression changes sign just once and in general $f(w)$ first decreases and then increases. Taking into account that $0 < s < 1$ and constraint from (1) we can conclude that for any $0 < w < 1$, function $f(w)$ is lower then v, which in turn is lower than 1. We can conclude that $f(w)$ crosses 45 degree line only once from above and thus there is a unique solution to (7).

Proof of Corollary 1

First lets identify the threshold cost $\bar{c}$, s.t. there is advertising in the equilibrium. Substituting $s^* = 0$ into (5) and solving for $e^{\frac{\lambda w}{2}}$ we get $e^{\frac{\lambda w}{2}} = \frac{c\lambda+v}{2\sqrt{c}}$. Substituting it into (7) we get $w = \frac{1}{2}\left(c\left(\lambda - \frac{4}{c\lambda+v}\right) + v\right)$. Substituting it back to (5) we get the following equation:

$$1 + \frac{c\lambda - 2\sqrt{c}e^{\frac{1}{4}\lambda\left(c\left(\lambda-\frac{4}{c\lambda+v}\right)+v\right)}}{v} = 0,$$

We can express threshold c as a share of v. Substituting $c = \alpha^2 v^2$ we get:

$$1+\alpha^2\lambda v-2\alpha e^{\frac{1}{4}\lambda v\left(1+\alpha^2\lambda v-\frac{4\alpha^2}{1+\alpha^2\lambda v}\right)}$$

If we reexpress λv as γ we get

$$\frac{1+\alpha^2\gamma}{\alpha}-2e^{\gamma\frac{(1+\alpha^2\gamma)^2-4\alpha^2}{4(1+\alpha^2\gamma)}}$$

One can show that the function is negative for any $\gamma>1$ only if $\alpha\geq 1$. Thus threshold level $\bar{c}$ equals v^2.

Lemma 2 *A variable* $q=\frac{e^{\frac{\lambda w}{2}}}{\sqrt{c}}$ *decreases in* c.

Rearranging the last expression and substituting $q=\frac{e^{\frac{\lambda w}{2}}}{\sqrt{c}}$ we get:

$$w=\frac{v}{2}-\frac{1}{2}\left(\frac{2}{q}-\frac{v}{\frac{2q}{\lambda}-1}\right) \tag{11}$$

Taking the derivative of q with respect to c we get:

$$\frac{e^{\frac{\lambda w}{2}}\left(c\lambda w'(c)-1\right)}{2c^{3/2}}$$

Thus if $w'(c)<0$ then q decreases in c. To consider the case when $w'(c)>0$ lets calculate the derivative $w'(c)$ using implicit function theorem. We get the following:

$$w'(c)=\frac{-c\lambda^2+4\sqrt{c}\lambda e^{\frac{\lambda w}{2}}+(\lambda v-4)e^{\lambda w}}{6c^{3/2}\lambda^2e^{\frac{\lambda w}{2}}-c^2\lambda^3+c\lambda(\lambda v-12)e^{\lambda w}+8\sqrt{c}e^{\frac{3\lambda w}{2}}} \tag{12}$$

Substituting it to the the expression for $q'(c)$ we get:

$$\frac{1}{c}\left(\lambda\left(\frac{1}{\frac{e^{\lambda w}}{c}}-\frac{\lambda v}{\left(\lambda-\frac{2e^{\frac{\lambda w}{2}}}{\sqrt{c}}\right)^2}\right)-\frac{2}{\frac{e^{\frac{\lambda w}{2}}}{\sqrt{c}}}\right)^{-1}$$

Substituting q we get that the previous expression is negative whenever the following holds:

$$-\frac{2q-\lambda}{q^2}-\frac{\lambda v}{(2q-\lambda)^2}>0$$

We know that $2q>\lambda$ and thus $q'(c)$ is always negative.

Lemma 3 *Variable* q *always increases in* λ*. Moreover,* $q<\frac{2\lambda}{4-\lambda v}$ *is true only for sufficiently high* λ*.*

Proof

The full derivative of $q = \frac{e^{\frac{\lambda w}{2}}}{\sqrt{c}}$ with respect to λ is

$$\frac{e^{\lambda w}\left(-\sqrt{c}\lambda(4w-v)e^{\frac{\lambda w}{2}} + c\lambda^2 w + 4we^{\lambda w}\right)}{6c^{3/2}\lambda^2 e^{\frac{\lambda w}{2}} - c^2\lambda^3 + c\lambda(\lambda v - 12)e^{\lambda w} + 8\sqrt{c}e^{\frac{3\lambda w}{2}}}$$

The denominator is positive. Substituting back $q = \frac{e^{\frac{\lambda w}{2}}}{\sqrt{c}}$ into the numerator we get $q\left(\lambda qv + 4qw(q-\lambda) + \lambda^2 w\right)$, which is higher than zero since $q > \lambda$. Thus q increases in λ. In particular, this implies that λw is increasing in λ.

Let's show that there is just one point of intersection of curves represented by q and $\frac{2\lambda}{4-\lambda v}$. Taking the derivative of q with respect to λ and substituting the derivative of $w'(\lambda)$ we get:

$$\frac{e^{\lambda w}\left(-\sqrt{c}\lambda(4w-v)e^{\frac{\lambda w}{2}} + c\lambda^2 w + 4we^{\lambda w}\right)}{6c^{3/2}\lambda^2 e^{\frac{\lambda w}{2}} - c^2\lambda^3 + c\lambda(\lambda v - 12)e^{\lambda w} + 8\sqrt{c}e^{\frac{3\lambda w}{2}}}$$

Substituting $q = \frac{e^{\frac{\lambda w}{2}}}{\sqrt{c}}$ we get:

$$\frac{q^2\left(4q^2 w - \lambda q(4w-v) + \lambda^2 w\right)}{8q^3 - \lambda^3 + \lambda q^2(\lambda v - 12) + 6\lambda^2 q} = \frac{q^2\left(4q^2 w - \lambda q(4w-v) + \lambda^2 w\right)}{(2q-\lambda)^3 + v\lambda^2 q^2}$$

The denominator is positive, since $q > \lambda$. We know that at the point of intersection $q = \frac{2\lambda}{4-\lambda v}$ substituting it to the expression we get

$$\frac{4\lambda^2 v(8 - \lambda v(2 - \lambda w))}{(4-\lambda v)^2\,(32 - 16\lambda - \lambda^3 v^2)}$$

One can show that the last expression for any w is lower than the derivative of $\frac{2\lambda}{4-\lambda v}$ with respect to λ, which is $\frac{8}{(4-\lambda v)^2}$. Thus we can conclude that whenever we have an intersection of two curves it should be that $\frac{2\lambda}{4-\lambda v}$ crosses q from below. Hence, there is at most one point of intersection, s.t. q intersects $\frac{2\lambda}{4-\lambda v}$ from below.

When $\lambda = 0$, q equals $\frac{1}{\sqrt{c}}$ and thus for sufficiently small λ, q is higher than $\frac{2\lambda}{4-\lambda v}$. When λ approaches $\frac{4}{v}$, expression $\frac{2\lambda}{4-\lambda v}$ approaches infinity, while q is always limited for any finite λ. Thus there is a unique intersection point of q and $\frac{2\lambda}{4-\lambda v}$, s.t. q intersects $\frac{2\lambda}{4-\lambda v}$ from below, and thus $q < \frac{2\lambda}{4-\lambda v}$ is true for sufficiently high λ.

Proof of Proposition 1

Sales in the case of complete and incomplete information

$$w^* = \frac{v}{2} - \frac{\sqrt{c}}{2}\left(\frac{2}{e^{\frac{\lambda w^*}{2}}} - \frac{\lambda v}{2e^{\frac{\lambda w^*}{2}} - \lambda\sqrt{c}}\right)$$

Sales in the case of incomplete information are higher than sales in the case of full information if the second term is positive. Substituting $q = \frac{e^{\frac{\lambda w}{2}}}{\sqrt{c}}$ to the second term we get

$$\frac{\sqrt{c}e^{-\frac{\lambda w}{2}}\left(2\sqrt{c}\lambda + \lambda v e^{\frac{\lambda w}{2}} - 4e^{\frac{\lambda w}{2}}\right)}{2(2e^{\frac{\lambda w}{2}} - \sqrt{c}\lambda)} = \frac{2\lambda + \lambda q v - 4q}{2q(2q-\lambda)}$$

The denominator is always positive. The numerator is positive if $\lambda v > 4$ or $\lambda v < 4$ and $q < \frac{2\lambda}{4-\lambda v}$. We know that q decreases in c, while $\frac{2\lambda}{4-\lambda v}$ does not depend on c. Thus sales will be higher than $\frac{v}{2}$ for sufficiently high c if there is q, such that $\lambda < q < \frac{2\lambda}{4-\lambda v}$. Rearranging we get that condition always holds for $2 \geq \lambda v \leq 4$.

We also know that q increases in λ. By Lemma 3 we know that there is a unique intersection point of q and $\frac{2\lambda}{4-\lambda v}$, s.t. $\frac{2\lambda}{4-\lambda v}$ intersects q from below. Thus we can conclude that for sufficiently high λ sales are higher than $\frac{v}{2}$, since there always be such λ that $2 \geq \lambda v$.

The fact that the optimal price is always lower than in the case of full information immediately follows from Lemma 1.

Proof of Proposition 2

s^* ***decreases in*** c

Taking the derivative of (5) and assuming that w is a function of c we get:

$$\frac{\lambda}{v} - \frac{e^{\frac{\lambda w}{2}}(c\lambda w'(c) + 1)}{v\sqrt{c}}$$

Substituting (12) we get:

$$-\frac{1}{\sqrt{c}v}\left(\frac{1}{2e^{\frac{\lambda w}{2}} - \sqrt{c}\lambda} + \frac{2e^{\lambda w}}{\sqrt{c}v\lambda^2 e^{\lambda w} + \left(2e^{\frac{\lambda w}{2}} - \sqrt{c}\lambda\right)^3 - 2e^{\lambda w}\left(2e^{\frac{\lambda w}{2}} - \sqrt{c}\lambda\right)}\right)^{-1}$$

We know that $2e^{\frac{\lambda w}{2}} - \sqrt{c}\lambda > 0$ and thus the derivative is negative if the second term in the brackets is positive.

Lets denote by $q = e^{\frac{\lambda w}{2}}$. From the previous analysis we know that s and P are meaningful when $\sqrt{c}\lambda < 2q - \sqrt{c}\lambda < \frac{v}{\sqrt{c}}$ and q is higher than 1, since $w \geq 0$. Thus the condition becomes $\max\{\sqrt{c}\lambda, 1\} \leq q \leq \frac{1}{2}\left(\sqrt{c}\lambda + \frac{v}{\sqrt{c}}\right)$.

Using q we can rewrite the condition in the following form $f(q) = -2q^2\left(2q - \sqrt{c}\lambda\right) + \sqrt{c}\lambda q^2(\lambda v) + \left(2q - \sqrt{c}\lambda\right)^3 > 0$. Taking the derivative of the last expression with respect to q we get $f'(q) = 2\left(3c\lambda^2 + \sqrt{c}\lambda q(\lambda v - 10) + 6q^2\right)$. The expression represents upward sloping parabola. Taking the derivative we can find the minimum $q_m = \sqrt{c}\lambda\frac{10-\lambda v}{12}$. Since $q_m < \sqrt{c}\lambda$, the range of admissible values for q is on the upward sloping part of the parabola. Thus if $\sqrt{c}\lambda > 1$ and $f'(\sqrt{c}\lambda) > 0$ or $\sqrt{c}\lambda \leq 1$ and $f'(1) > 0$ we can conclude that $f'(q)$ is positive on the whole range.

Substituting $\sqrt{c}\lambda$ to $f()$ we get $2c\lambda^2(\lambda v - 1)$, which is positive if $\lambda v > 1$. Note that $\sqrt{c}\lambda > 1$ and thus $\lambda > \frac{1}{\sqrt{c}}$ and thus $1 < \sqrt{c}\lambda < \frac{v}{\sqrt{c}} < \lambda v$. Assume now that $\sqrt{c}\lambda < 1$, substituting $q = 1$ we get $2\left(3c\lambda^2 + \sqrt{c}\lambda(\lambda v - 10) + 6\right)$, which is positive, since $\sqrt{c}\lambda < 2 - \sqrt{c}\lambda < \frac{v}{\sqrt{c}}$.

Going back to $f()$ we should check that if $\sqrt{c}\lambda > 1$ then $f(\sqrt{c}\lambda) = c^{3/2}\lambda^3(\lambda v - 1)$ is higher than zero. We already have seen that if $\sqrt{c}\lambda > 1$ then $\lambda v > 1$ and thus $f(\sqrt{c}\lambda) > 0$. Substituting $q = 1$ we get $\sqrt{c}\lambda\left(\lambda\left(-c\lambda + 6\sqrt{c} + v\right) - 10\right) + 4$, which is greater than zero.

P^* decreases in c

Taking into account that q always decreases in c it is easy to show that the optimal price represented by (6) always decreases in c too.

Number of informed consumers $\frac{w^*}{v-P^*}$ decreases in c

We know that in the equilibrium w^* consumers buy the product. Each of the informed consumers buys the product with probability $v - P^*$ and thus the number of informed consumers is the following:

$$\frac{w^*}{v - P^*} = \frac{\lambda + q(qv - 2)}{q^2 v}$$

Taking the derivative with respect to q we get $\frac{2(q-\lambda)}{q^3 v}$ which is positive, since by (10) q is always greater than λ. Thus the number of informed consumers increases in q and consequently always falls in c.

Proof of Proposition 3

In general w^* first decreases and then increases in c

Taking the derivative of RHS of (11) with respect to q we get:

$$\frac{1}{q^2} - \frac{\lambda v}{(2q - \lambda)^2} \tag{13}$$

The expression is positive if $(4 - \lambda v)q^2 - 4\lambda q + \lambda^2 > 0$. If $\lambda v < 4$ the last expression is upward sloping parabola and it is positive if $q < q_1 = \frac{\lambda}{2+\sqrt{\lambda v}}$ or $q > q_2 = \frac{\lambda}{2-\sqrt{\lambda v}}$. Substituting q into (10) we get $\lambda < q < \frac{1}{2}\left(\lambda + \frac{v}{c}\right)$. Thus q is always greater than q_1. We know that q decreases in c and if $q > q_2$ then there are values of c such that w decreases in c. This happens if $\frac{\lambda}{2-\sqrt{\lambda v}} < \frac{1}{2}\left(\lambda + \frac{v}{c}\right)$, which reduces to $2\sqrt{\frac{v}{\lambda}} > v + c\lambda$. Since we are interested in the existence we substitute $c = 0$ and obtain $\lambda v < 4$.

When q is lower than q_2 expression (13) is negative and w increases in c. Actually, the increasing behavior appears when $\lambda < \frac{\lambda}{2-\sqrt{\lambda v}}$, which holds if $\lambda v > 1$.

When $\lambda v > 4$ we have downwards sloping parabola. In this case q_2 becomes negative and λ is higher than $\frac{\lambda}{2+\sqrt{\lambda v}}$. Thus expression (13) is negative and w increases on the whole range in c.

Proof of Proposition 4

Consumer surplus $w^* \frac{v-P^*}{2}$ ***is non monotone in*** c

Substituting expressions for w^* and P^* in terms of q to $CS = w^* \frac{v-P^*}{2}$ we get:

$$\frac{v(\lambda + q(qv - 2))}{2(2q - \lambda)^2}$$

Deriving it with respect to q we get:

$$-\frac{v(\lambda + q(\lambda v - 2))}{(2q - \lambda)^3}$$

We know that the denominator is positive and thus the whole expression is negative if $\lambda v > 2$ or $q < \frac{\lambda}{2-\lambda v}$. By condition (10) we know that $\lambda < q < \frac{1}{2}\left(\frac{v}{c} + \lambda\right)$. Thus there is a region where the derivative is negative if $\lambda v > 1$. Taking into account that the derivative of q with respect to c is negative we can conclude that if $\lambda v > 1$ then there is a region where CS increases in c.

Proof of Proposition 5

In general social welfare first decreases in c ***and then increases.***

Taking the derivative of the expression for social welfare with respect to q we get:

$$-\frac{v(q - \lambda)\left((\lambda v - 6)q^2 + 5\lambda q - \lambda^2\right)}{q^2(2q - \lambda)^3}$$

The sign of the derivative depends on sign of the following expression $(\lambda v - 6)q^2 + 5\lambda q - \lambda^2$, since we know that $q > \lambda$. If $\lambda v > 6$ it is always positive and thus SW decreases in q and consequently increases in c. If however $\lambda v < 6$ we have downward sloping parabola with roots $q_1 = \frac{5\lambda - \lambda\sqrt{1+4\lambda v}}{2(6-\lambda v)}$ and $q_2 = \frac{5\lambda + \lambda\sqrt{1+4\lambda v}}{2(6-\lambda v)}$. We can show that if $\lambda v < 6$ the first root is always less than λ and thus does not belong to admissible parameter range. The second root is higher than λ if $\lambda v > 2$ and is lower than $\frac{1}{2}\left(\frac{v}{c} + \lambda\right)$ if $c < \frac{1-\lambda v+\sqrt{1+4\lambda v}}{\lambda^2}$.

Summing up if $\lambda v < 2$ then SW always decreases in c. If $2 < \lambda v < 6$ then SW first decreases in c up to the point where $c = \frac{1-\lambda v+\sqrt{1+4\lambda v}}{\lambda^2}$ and then increases. If $\lambda v > 6$ then SW always increases in c.

Proof of Proposition 6

The amount of advertisement s^* first increases, but then decreases in λ

Taking the full derivative of solution for s we get $\frac{c-\sqrt{c}e^{\frac{1}{2}\lambda w(\lambda)}(\lambda w'(\lambda)+w(\lambda))}{v}$. Substituting the derivative of $w(\lambda)$ with respect to λ we get the following expression:

$$c+\frac{-2c^{3/2}\lambda^2 we^{\lambda w}+2c\lambda(4w-v)e^{\frac{3\lambda w}{2}}-8\sqrt{c}we^{2\lambda w}}{8e^{\frac{3\lambda w}{2}}-\lambda^3c^{3/2}+\sqrt{c}\lambda(\lambda v-12)e^{\lambda w}+6c\lambda^2e^{\frac{\lambda w}{2}}}$$

Substituting $q=\frac{e^{\frac{\lambda w}{2}}}{\sqrt{c}}$ we get:

$$c\left(1-\frac{q^2\left(8q^2w-2\lambda q(4w-v)+2\lambda^2w\right)}{(2q-\lambda)^3+v\lambda^2q^2}\right)$$

Expressing equation for w in terms of q we get:

$$w=\frac{1}{2}\left(v-\left(\frac{2}{q}-\frac{\lambda v}{2q-\lambda}\right)\right) \tag{14}$$

Substituting last equation to the derivative we get:

$$c\left(1+\frac{2q\left(\lambda^2-2q^3v+4q^2-4\lambda q\right)}{(2q-\lambda)^3+v\lambda^2q^2}\right) \tag{15}$$

We know that $\lambda<q<\frac{1}{2}\left(\frac{v}{c}+\lambda\right)$. When λ equals 0, $q=\frac{1}{\sqrt{c}}$. Substituting it into the derivative we get $2-\frac{v}{2\sqrt{c}}$, which is positive if $c>\frac{v^2}{16}$. Thus if $c>\frac{v^2}{16}$ the optimal advertising first increases in λ and otherwise decreases in λ.

Expression in the brackets in (15) is positive if $\lambda^2-2q^3v-q^2(\lambda v-8)-6\lambda q>0$ and is negative otherwise. The expression decreases in λ for all values. Taking the derivative with respect to q we get quadratic expression in q, $-2\left(3\lambda+3q^2v+q(\lambda v-8)\right)$. One can show that if $\lambda v\geq\frac{5}{4}$ then the expression is negative and thus the derivative decreases in q. We know that q increases in λ. Thus when $\lambda\geq\frac{5}{4v}$ we can conclude that the amount of advertising decreases in λ

Lemma 4 *If there is no advertising in the equilibrium, the optimal price P^* always increases in λ*

Proof

Substituting into equation for the optimal price $s^*=0$ we get:

$$\hat{P}^*=\frac{\lambda v+\sqrt{e^{\lambda w^*}\left(e^{\lambda w^*}-\lambda v\right)}-e^{\lambda w^*}}{\lambda}=\sqrt{\frac{e^{\lambda w^*}}{\lambda}\left(\frac{e^{\lambda w^*}}{\lambda}-v\right)}-\left(\frac{e^{\lambda w^*}}{\lambda}-v\right) \tag{16}$$

And equation for sales becomes:

$$\lambda w^* = e^{-\lambda w^*}\left(e^{\lambda w^*}-1\right)\left(e^{\lambda w^*}-\sqrt{e^{\lambda w^*}\left(e^{\lambda w^*}-\lambda v\right)}\right) \tag{17}$$

Note that for existence of the solution $e^{\lambda w^*}$ should be higher than λv, thus the smallest possible value of w^* is $\frac{\ln(\lambda v)}{\lambda}$. Substituting it to (17) we get $\lambda v < e^{\lambda v - 1}$ which is always true. Thus the curve on the left hand side is above 45 degree line at $w = \frac{\ln(\lambda v)}{\lambda}$. The maximal value for w is v. Substituting $y = \lambda w = \lambda v$ we get $y(e^y+1) - 2e^y\left(e^y - \sqrt{e^y(e^y-y)}\right)$. Subtracting y and taking out common factor we get $y - 2\left(e^y - \sqrt{e^y(e^y-y)}\right)$, which is lower than 0, and thus the curve represented by the RHS of (17) is below 45 degree line.

Substituting $x = \lambda w^*$ and $y = \lambda v$ into (17) we get $x = e^{-x}\left(e^x-1\right)\left(e^x - \sqrt{e^x\left(e^x-y\right)}\right)$. The derivative of the right hand side of the equation with respect to x is:

$$\frac{e^x\left(2\sqrt{e^x\left(e^x-y\right)}-2e^x+y\right)+y}{2\sqrt{e^x\left(e^x-y\right)}}$$

The denominator is positive since $1 \leq y \leq e^x$. Solving equation (17) for y we get $y = \frac{e^x(-x+2e^x-2)x}{(e^x-1)^2}$. Substituting it to the numerator we get:

$$-\frac{xe^x(2(1-e^x)+x(1+e^x))}{(1-e^x)^2},$$

which is negative for any $x > 0$. This implies that if the curve crosses 45 degree line it should cross it from above. Thus the solution exists and it is unique.

Substituting $z = \frac{e^{\lambda w^*}}{\lambda}$ into P^* and taking the derivative with respect to z we get:

$$\frac{z-\frac{1}{2}v}{\sqrt{z(z-v)}}-1$$

The denominator is positive since $\lambda z = e^{\lambda w^*} > \lambda v$. It is easy to see that the expression is positive too. Thus we proved that P^* increases in $z = \frac{e^{\lambda w^*}}{\lambda}$. Taking the full derivative of $\frac{e^{\lambda w^*}}{\lambda}$ with respect to λ we get:

$$\frac{2e^{\lambda w^*}\left(e^{\lambda w^*}-\lambda v-\lambda w^*\sqrt{e^{\lambda w^*}\left(e^{\lambda w^*}-\lambda v\right)}\right)}{\lambda^2\left(\lambda v+e^{\lambda w^*}\left(\lambda v+2\sqrt{e^{\lambda w^*}\left(e^{\lambda w^*}-\lambda v\right)}-2e^{\lambda w^*}\right)-2\sqrt{e^{\lambda w^*}\left(e^{\lambda w^*}-\lambda v\right)}\right)}$$

Substituting λw^* from (17) into the expression in the brackets in the numerator we get:

$$e^{\lambda w^*}-\lambda v-e^{-\lambda w^*}\left(e^{\lambda w^*}-1\right)\left(e^{\lambda w^*}-\sqrt{e^{\lambda w^*}\left(e^{\lambda w^*}-\lambda v\right)}\right)\sqrt{e^{\lambda w^*}\left(e^{\lambda w^*}-\lambda v\right)}=$$

$$= \frac{\sqrt{e^{\lambda w^*} - \lambda v}}{\sqrt{e^{\lambda w^*}}} \left(\sqrt{e^{\lambda w^*} \left(e^{\lambda w^*} - \lambda v\right)} - \left(e^{\lambda w^*} - 1\right) \left(e^{\lambda w^*} - \sqrt{e^{\lambda w^*} \left(e^{\lambda w^*} - \lambda v\right)} \right) \right)$$

In the brackets we get a quadratic expression in $e^{\lambda w^*}$ which is negative if $\lambda v \geq 2$ or $e^{\lambda w^*} < \frac{1}{2-\lambda v}$. Substituting $e^{\lambda w^*} = \frac{1}{2-\lambda v}$ to the right hand side of (17) we get $\lambda w^* = \lambda v - 1$ or equivalently $e^{\lambda w^*} = e^{\lambda v - 1}$. Since $e^{\lambda v - 1} \leq \frac{1}{2-\lambda v}$, w^* is s.t. $e^{\lambda w^*} < \frac{1}{2-\lambda v}$, which implies that the optimal w^* is lower than $\frac{1}{2-\lambda v}$. Hence, the derivative of the price with respect to λ is positive.

Proof of Proposition 7

Price P^* decreases in λ

Lets consider term $\frac{q}{\lambda} = \frac{e^{\frac{\lambda w}{2}}}{\sqrt{c}\lambda}$. Taking full derivative of it with respect to λ we get:

$$\frac{e^{\frac{\lambda w}{2}} \left(2e^{\frac{\lambda w}{2}} - \lambda\sqrt{c}\right)^2 \left(\lambda\sqrt{c} + e^{\frac{\lambda w}{2}}(\lambda w - 2)\right)}{\lambda^2\sqrt{c}\left(\sqrt{c}\lambda(\lambda v - 12)e^{\lambda w} + 6c\lambda^2 e^{\frac{\lambda w}{2}} + 8e^{\frac{3\lambda w}{2}} - \lambda^3 c^{3/2}\right)}$$

As we know the denominator is positive. The numerator is positive whenever

$$\sqrt{c} + \frac{e^{\frac{\lambda w}{2}}}{\lambda}(\lambda w - 2) > 0 \tag{18}$$

When λ is sufficiently close to 0, the expression is negative and thus $\frac{e^{\frac{\lambda w}{2}}}{\lambda}$ falls in λ, while by Lemma 3, λw increases in λ. Assume that there is $\hat{\lambda}$, such that the expression becomes positive. Assume further that exists $\tilde{\lambda} > \hat{\lambda}$, such that the expression becomes zero again. In this case we know that $\frac{e^{\frac{\lambda w}{2}}}{\lambda}$ does not grow in λ, while λw continues to grow and thus for any sufficiently small $\epsilon > 0$ term $\frac{e^{\frac{(\lambda+\epsilon)w}{2}}}{\sqrt{c}(\lambda+\epsilon)}$ should increase in ϵ. Taking into account the continuity of all involved functions we can conclude that for any $\lambda > \hat{\lambda}$ the term is non decreasing in λ, which implies that the optimal price is also non decreasing in λ. Thus we have shown that in general the optimal price first decreases, but than increases.

Now let's check the conditions under which we have each part of the curve. For sufficiently small λ we know that (18) is negative and thus price decreases in λ. Thus the price first decreases and then may increase. We also know that advertising first increases and then decreases. Thus to check whether there is increasing part of the curve we substitute solution for $s = 0$ to (18). Thus substituting (**??**) we obtain:

$$\frac{(v + \lambda c)(\lambda(v + \lambda c) - 4)}{4\lambda\sqrt{c}}$$

The expression is positive whenever $\lambda(v + \lambda c) > 4$.

Taking the full derivative of solution for P^* we get $\frac{\sqrt{c} v e^{\frac{\lambda w}{2}}\left(\lambda^2 w'(\lambda)+\lambda w-2\right)}{2\left(2e^{\frac{\lambda w}{2}}-\sqrt{c}\lambda\right)^2}$. Substituting $w'(\lambda)$ we get:

$$\frac{\sqrt{c} v e^{\frac{\lambda w}{2}}(\sqrt{c}\lambda + e^{\frac{\lambda w}{2}}(\lambda w - 2))}{\sqrt{c}\lambda(\lambda v - 12)e^{\lambda w} + 6c\lambda^2 e^{\frac{\lambda w}{2}} + 8e^{\frac{3\lambda w}{2}} - \lambda^3 c^{3/2}}$$

Substituting $q = \frac{e^{\frac{\lambda w}{2}}}{\sqrt{c}}$ we obtain:

$$\frac{q(\lambda v - qv(2 - \lambda w))}{(2q - \lambda)^3 + v\lambda^2 q^2} \tag{19}$$

The denominator is positive. Substituting (14) into the numerator of (19) we get:

$$\frac{qv\lambda(2 + \frac{q}{\lambda}(\lambda v - 4))}{2q - \lambda} \tag{20}$$

The numerator of (19) is negative iff $\lambda v < 4$ and $q > \frac{2\lambda}{4-\lambda v}$. Note that for λ sufficiently close to 0, q is higher or equal to $\frac{1}{\sqrt{c}}$. Thus for a sufficiently small λ the optimal price decreases in λ. As λ grows further two things may happen. It may approach $\frac{4}{v}$, in which case we know that the optimal price starts to increase in λ. Another possibility is that at some point the optimal s^* becomes zero and we get the corner solution with $s^* = 0$. However, by Lemma 4 we know that when there is no advertising P^* increases in λ if $\lambda v > 1$. Thus if $\lambda v > 1$ the price is non-monotone in λ.

Proof of Proposition 8

For sufficiently small λ sales increase in λ, but when λ becomes sufficiently high sales decrease in λ

The derivative of w with respect to λ is

$$\frac{c^{3/2}\lambda^2 w + \sqrt{c} e^{\lambda w}(v(2 - \lambda w) + 4w) - 4c\lambda w e^{\frac{\lambda w}{2}}}{\sqrt{c}\lambda(\lambda v - 12)e^{\lambda w} - \lambda^3 c^{3/2} + 6c\lambda^2 e^{\frac{\lambda w}{2}} + 8e^{\frac{3\lambda w}{2}}}$$

The denominator is positive. Substituting $\lambda = 0$ we get $\frac{1}{8}\sqrt{c}(2v+4w)$, which is positive and thus for sufficiently small λ, sales increase in λ.

Substituting $q = \frac{e^{\frac{\lambda w}{2}}}{\sqrt{c}}$ we get

$$\frac{\lambda^2 w + q^2 v(2 - \lambda w) + 4qw(q - \lambda)}{(2q - \lambda)^3 + v\lambda^2 q^2}$$

Substituting maximal value for q, $q = \frac{1}{2}\left(\frac{v}{c} + \lambda\right)$ we get

$$c\frac{4vw+(c\lambda+v)^2(2-\lambda w)}{c^3\lambda^4+2c^2\lambda^3v+4v^2+cv^2\lambda^2}$$

Note that λw increases in λ and we can choose arbitrarily large λ for sufficiently small c. Thus the expression is negative for sufficiently high λ, since w is limited from above by 1.

For sufficiently small λ consumer surplus increases in λ, but when λ becomes sufficiently high consumer surplus decreases in λ

The consumer surplus is given by $w^*\frac{v-P}{2}$. We know that for sufficiently small λ sales increase in λ and by Propositions 7 we know that price decreases in λ. Thus for sufficiently small λ consumer surplus increases in λ. We also know that when λ becomes sufficiently high sales decrease in λ, while price increases in λ. Thus for sufficiently high λ consumer surplus decreases in λ.